Journal of Combinatorial Theory, Series B **77**, 150–161 (1999)
Article ID jctb.1999.1918, available online at http://www.idealibrary.com on IDEAL®

On 3-Connected Plane Graphs without Triangular Faces

J. Harant

Department of Mathematics, Technical University of Ilmenau, D-98684 Ilmenau, Germany
E-mail: harant@mathematik.tu-ilmenau.de

and

S. Jendrol'* and M. Tkáč†

Department of Geometry and Algebra, P. J. Šafárik University, Jesenná 5, 041 54 Košice, Slovak Republic

We prove that each polyhedral triangular face free map G on a compact 2-dimensional manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ contains a k-path, i.e., a path on k vertices, such that each vertex of this path has, in G, degree at most $(5/2)\,k$ if $\mathbb{M}$ is a sphere $\mathbb{S}_0$ and at most $(k/2)\lfloor(5+\sqrt{49-24\chi(\mathbb{M})})/2\rfloor$ if $\mathbb{M}\neq\mathbb{S}_0$ or does not contain any k-path. We show that for even k this bound is best possible. Moreover, we show that for any graph other than a path no similar estimation exists. © 1999 Academic Press

Key Words: planar graph; polyhedral map; light subgraph; triangle-free graph; path; compact 2-dimensional manifold; spanning subgraph.

1. INTRODUCTION

Throughout this paper we shall consider connected graphs without loops or multiple edges. Let P_r denote a path on r vertices (an r-path in what follows). For graphs H and G, $G\cong H$ denotes that the graphs H and G are isomorphic. The standard notation $\Delta(G)$ stands for the maximum degree of a graph G. For a vertex X of a graph G $\deg_G(X)$ denotes the degree of X in G.

Let $\mathscr{H}$ be a family of graphs and let H be a graph which is isomorphic to a proper subgraph of at least one member of $\mathscr{H}$. Let $\varphi(H,\mathscr{H})$ be the smallest integer with the property that every graph $G\in\mathscr{H}$ which has a subgraph isomorphic with H also contains a subgraph K, $K\cong H$, such that, for every vertex $A\in V(K)$,

$$\deg_G(A)\leqslant\varphi(H,\mathscr{H}).$$

0095-8956/99 $30.00

If such a $\varphi(H, \mathscr{H})$ does not exist we write $\varphi(H, \mathscr{H}) = +\infty$. If $\varphi(H, \mathscr{H}) < +\infty$ we call the graph H *light in the family* $\mathscr{H}$.

Let $\mathscr{P}(\delta, \rho)$ be the family of all 3-connected plane graphs with minimum vertex degree at least δ and minimum face size at least ρ. Let $\varphi(\delta, \rho; H) = \varphi(H, \mathscr{P}(\delta, \rho))$.

Kotzig [Kot1] has proved that each graph $G \in \mathscr{P}(3, 3)$ contains an edge AB with $\deg_G(A) + \deg_G(B) \leqslant 13$. From the classical result of Lebesgue [Lebe] it follows that each graph $H \in \mathscr{P}(3, 4)$ contains an edge AB such that $(\deg_H(A), \deg_H(B)) \in \{(3, 3), (3, 4), (3, 5)\}$.

Fabrici and Jendrol' [FaJe] proved

THEOREM 1. (i) *H is light in $\mathscr{P}(3, 3)$ if and only if $H \cong P_k$, $k \geqslant 1$.*

(ii) $\varphi(3, 3; P_k) = 5k$, $k \geqslant 1$.

Fabrici *et al.* [FHJW] have proved.

THEOREM 2. (i) *H is light in $\mathscr{P}(4, 3)$ if and only if $H \cong P_k$, $k \geqslant 1$.*

(ii) $\varphi(4, 3; P_k) = 5k - 7$, $k \geqslant 8$.

The first main result of this paper is the following.

THEOREM 3. (i) *H is light in $\mathscr{P}(3, 4)$ if and only if $H \cong P_k$, $k \geqslant 1$.*

(ii) $5\lfloor k/2 \rfloor \leqslant \varphi(3, 4; P_k) \leqslant \frac{5}{2}k$, $k \geqslant 2$.

Notice that for even k we have the exact value $\varphi(3, 4; P_k)$. We believe that for odd k the precise value of $\varphi(3, 4; P_k)$ is $5\lfloor k/2 \rfloor$.

The rest of the paper is organized as follows. In Section 2 we give a proof of Theorem 3. In Section 3 we give an application of Theorem 3. Section 4 is devoted to our second main result, an analogue of Theorem 3 for polyhedral maps on compact 2-manifolds. In Section 5 we will discuss related problems.

2. PROOF OF THEOREM 3

The proof of the theorem for $k = 1, 2$ follows from [Lebe]; for $k \geqslant 3$ it is a consequence of the following three lemmas.

LEMMA 1. *Each graph $G \in \mathscr{P}(3, 4)$ with $\Delta(G) \geqslant k$, $k \geqslant 3$, contains a path P_k, a path on k vertices, such that each vertex of this path has, in G, degree at most $\frac{5}{2}k$.*

Proof. Suppose there is a counterexample G on n-vertices and having maximum number of edges among all counterexamples on n vertices. Let us call vertices of degree $>\frac{5}{2}k$ *major* vertices; the other ones we will call *minor*.

PROPERTY 1. *Each major vertex of G is incident only with quadrangular and/or pentagonal faces.*

Proof. If a major vertex B is incident with an r-face α, $r \geqslant 6$, then we can insert a diagonal joining the vertex B with a vertex D which is at distance 3 from B on the boundary of α. The result is a new graph G' with one edge more than G, which is also a counterexample, a contradiction. ∎

Let $G_0 = G, G_1, ..., G_p = G^*$ be a sequence of plane graphs defined as follows: If G_i, $i = 0, 1, ...$, is a plane 3-connected graph having an r-gonal face α, $r \geqslant 4$, incident with two non-adjacent major vertices A and B we insert a diagonal $d = AB$ into α joining the vertices A and B. The result is a plane 3-connected graph $G_{i+1} = G_i + d$. If G_i does not contain any face α having the above-mentioned property we put $i = p$ and $G^* = G_p$.

Let $M = M(G^*)$ be the subgraph of G^* induced by the major vertices of G^*. Because G^* is plane there is, in M, a vertex A such that

$$\deg_M(A) \leqslant 5.$$

On the other side

$$\deg_{G^*}(A) \geqslant \deg_G(A) \geqslant \tfrac{5}{2}k + \tfrac{1}{2}.$$

This means that on the faces incident with A in G there are at least $2(\frac{5}{2}k + \frac{1}{2}) = 5k + 1$ vertices different from A which form a cycle $\mathscr{C}$. Among these vertices there are at most 5 major neighbours $A_1, A_2, ..., A_q$ $(q \leqslant 5)$ of A (consecutive around A) in G^*. Let B be a major vertex on $\mathscr{C}$ which is not adjacent to A in G^*. We may assume that B is between A_1 and A_2. Then A and B belong to the boundary of a common face F of G and F contains two further major vertices D and E which are connected by a diagonal in G^*. Since, by Property 1, F is at most a 5-gon and D or E is a neighbour of A in G (and consequently in G^*, too). Without loss of generality let $E = A_2$. If $A_1 \neq D$ then F is a 5-gon $[A, Z, D, B, A_2]$. If Z is minor in G^* there must be an edge AD and then $A_1 = D$, a contradiction. Thus, Z is a major, $Z = A_1$, and G^* contains the diagonal A_1A_2. If $A_1 = D$ then F contains the major vertices A, $A_1 = D$, B, $A_2 = E$ and at most one minor vertex is between A_1 and A_2 on $\mathscr{C}$. Thus, each component of $\mathscr{C} - \{A_1, A_2, ..., A_g\}$ that contains a major vertex contains no more than 2 vertices altogether.

Suppose no component contains a major vertex. Then there are at least $5k+1-5=5k-4$ minor vertices in at most 5 components, and at least one component is a k-path of minor vertices. On the other hand, suppose one component contains a major vertex, so it has at most 2 vertices altogether. Then in the remaining components there are at least $5k+1-5-2=5k-6 \geqslant 4k-3$ minor vertices in at most 4 components, and so some component contains at least k vertices. Since $k \geqslant 3$ this component consists of minor vertices only and gives the required path. ∎

LEMMA 2. *There is a graph $G \in \mathcal{P}(3, 4)$ in which every path on k-vertices contains a vertex of degree at least $5\lfloor k/2 \rfloor$, $k \geqslant 2$.*

Proof. For $k=2$ or 3 consider G to be dual to the Archimedian (3, 5, 3, 5)-solid (i.e., the icosi-dodecahedron; see [Crom]). For $k \geqslant 4$ we prove the existence of a 3-connected plane triangle-free graph in which each k-path contains a vertex of degree at least $5\lfloor k/2 \rfloor$. Our construction starts with a plane 3-connected triangulation H of minimum degree 5. We shall distinguish two cases according to the remainder on dividing k by 4. Let $k=4t+z$, $t=1, 2, ..., 0 \leqslant z \leqslant 3$.

Case 1: $z=0$ or 1. Each triangle $[A_0 B_0 C_0]$ of H is decomposed into three triangles faces $[A_0 X_1 B_0]$, $[B_0 Y_1 C_0]$, $[A_0 Z_1 C_0]$ and into the following three groups of quadrangular faces

(i) $[A_0 X_1 A_1 X_2], [A_0 X_2 A_2 X_3], ..., [A_0 X_{t-1} A_{t-1} X_t]$;
$[A_0 X'_1 C_1 X'_2], [A_0 X'_2 C_2 X'_3], ..., [A_0 X'_{t-1} C_{t-1} X'_t]$;
$[A_1 X_2 A_2 Y'_2], [A_2 X_3 A_3 Y'_3], ..., [A_{t-1} X_t A_t Y'_t]$ and
$[A_0 X_t A_t X'_t]$.
(Note that in the following $X_1 = Y'_1$, $Y_1 = Z'_1$, $Z_1 = X'_1$ and
$A_t = B_t = C_t$, i.e., the vertices A_t, B_t and C_t coincide)

(ii) $[B_0 Y'_1 A_1 Y'_2], [B_0 Y'_2 A_2 Y'_3], ..., [B_0 Y'_{t-1} A_{t-1} Y'_t]$;
$[B_0 Y_1 B_1 Y_2], [B_0 Y_2 B_2 Y_3], ..., [B_0 Y_{t-1} B_{t-1} Y_t]$;
$[B_1 Y_2 B_2 Z'_2], [B_2 Y_3 B_3 Z'_3], ..., [B_{t-1} Y_t B_t Z'_t]$ and $[B_0 Y_t B_t Y'_t]$.

(iii) $[C_0 Z_1 C_1 Z_2], [C_0 Z_2 C_2 Z_3], ..., [C_0 Z_{t-1} C_{t-1} Z_t]$;
$[C_0 Z'_1 B_1 Z'_2], [C_0 Z'_2 B_2 Z'_3], ..., [C_0 Z_{t-1} B_{t-1} Z'_t]$;
$[C_1 X'_2 C_2 Z_2], [C_2 X'_3 C_3 Z_3], ..., [C_{t-1} X'_t C_t Z_t]$ and
$[C_0 Z_t C_t Z'_t]$.

The construction finishes by deleting all original edges A_0B_0, A_0C_0 and B_0C_0 of H. The result is a 3-connected plane graph G having the required properties. See Fig. 1 for an illustration of a decomposition of a triangle $A_0B_0C_0$ when $k=12$ or $k=13$.

Case 2: $z=2$ or 3. The construction is analogous the above one. The only difference is this case is that the vertices A_t, B_t and C_t do not coincide but they are mutually distinct and have a common neighbour D which is also adjacent to the vertices A_0, B_0 and C_0. Instead of the three quadrangles $[A_0X_tA_tX'_t]$, $[B_0Y_tB_tY'_t]$, and $[C_0Z_tC_tZ'_t]$ we have now six quadrangles $[A_0X_tA_tD]$, $[A_0DC_tX'_t]$, $[B_0Y_tB_tD]$, $[B_0DA_tY'_t]$, $[C_0Z'_tB_tD]$, and $[C_0DC_kZ_k]$. ∎

LEMMA 3. *For every connected plane graph H, $H \neq P_k$, $k \geqslant 1$, and every integer $m \geqslant 3$ there exists a graph $G \in \mathscr{P}(3,4)$ such that each copy of H in G contains a vertex A with* $\deg_G(A) \geqslant m$.

Proof. First the graph H is triangulated into a plane triangulation T with $V(T)=V(H)$. Then into each triangular face $[A_{1,0}A_{2,0}A_{3,0}]$ of T we insert three $(2m+2)$-paths $[A_{i,0}A_{i,1}A_{i,2}, ..., A_{i,2m+1}]$, and add edges $A_{i,0}A_{i+1,2p}$ and $A_{i,2m+1}A_{i-1,2p-1}$ for $p=1, 2, ..., m$ and $i=1, 2, 3$. (Note that the first indices are taken modulo 3). It is easy to check that the plane graph G obtained has the required properties. ∎

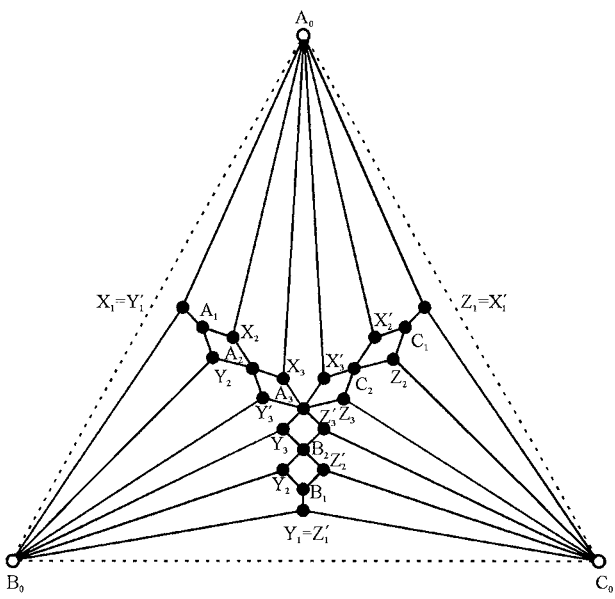

FIGURE 1

3. ON 2-CONNECTED SPANNING SUBGRAPHS OF PLANAR 3-CONNECTED GRAPHS

In [Bar1] Barnette proved that every planar 3-connected graph has a spanning tree of maximum degree at most 3. In [Bar2] the same author considers the following problem: What is the minimum number d_0 such that every planar 3-connected graph has a spanning 2-connected subgraph of maximum degree at most d_0? He proves that $6 \leqslant d_0 \leqslant 15$. Gao [Gao] improves this result to $d_0 = 6$. Applying Theorem 3 with $k = 4$ we provide a simple proof of the following.

THEOREM 4. *Every planar* 3-*connected graph* G *has a* 2-*connected spanning subgraph of maximum degree at most* 10.

For the sake of completeness we repeat the most important definitions. A graph is *k-connected* provided that between any two of its vertices there are k paths meeting only at their endpoints. An edge of a planar 3-connected graph G is *shrinkable* if shrinking it to a vertex produces a new planar 3-connected graph (when we shrink an edge, resulting multiple edges bounding 2-sided faces are coalesced). An edge is *removable* if removing it produces a new planar 3-connected graph. (When we remove an edge, two edges at any resulting 2-valent vertex are coalesced into a single edge). We shall say that a subgraph H of a graph G *spans* G if H contains every vertex of G. A planar 3-connected graph G is *coverable* provided that it can be spanned by a 2-connected subgraph H of G of maximum degree at most 10.

We shall use the following lemma of Barnette [Bar2]:

LEMMA 4. *Let* G *be a planar* 3-*connected graph without triangular faces. Then any edge* e *of* G *is either shrinkable or removable.*

The proof of the following lemma is analogous to that of [Bar2, Lemma 5]. The only difference is the definition of coverability by the number 15 instead of 10. We omit the proof.

LEMMA 5. *Let* G *be a planar* 3-*connected graph with* n *edges and a triangular face. If every planar* 3-*connected graph with less than* n *edges is coverable then* G *is coverable.*

Proof of Theorem 4. Our proof is by induction on the number of edges of G. If G has minimum number of edges then it is the graph of the tetrahedron and the theorem clearly holds. We assume the theorem to be true for all graphs with at most n edges and suppose G has $n + 1$ edges.

If G has triangular faces then Lemma 5 shows that G is coverable. If G does not have triangular faces then, by Theorem 3, G has a 4-path $P_4 = VXYZ$ having all vertices of degree $\leqslant 10$. Let e be the edge XY of P_4. By Lemma 1, e is either shrinkable or removable. We treat two cases.

Case 1: *e is shrinkable.* We shrink e to a vertex W producing a graph G' with fewer edges which is coverable by a subgraph H'. Now we split the vertex W to produce the original graph G. This induces a splitting in H'. Let K be the graph produced from H' by this splitting. All vertices of K remain of degree at most 10.

Subcase 1.1: *No edge of K meets X.* Therefore, in K, $\deg(V) \leqslant 9$ and $\deg(Y) \leqslant 9$ because both V and Y are of degree $\leqslant 10$ in G. If we add the edges VX and XY to K we obtain a 2-connected spanning subgraph H of G whose maximum degree is at most 10.

Subcase 1.2: *Edges of K meet both X and Y.* In this case, in K, $\deg(X) \leqslant 9$ and $\deg(Y) \leqslant 9$. If we add the edge XY to K we obtain a required spanning subgraph H of G.

Case 2: *e is removable.* We remove e producing a graph G' with fewer edges which by induction, is coverable by a subgraph H'. We return the edge e and consider H' as a subgraph of G.

Subcase 2.1: *X (or Y or both) is of degree* 3 *in G.* In this case we may assume that H' does not meet X (or Y or both) and, in H', $\deg(V) \leqslant 9$, $\deg(Y) \leqslant 9$ (or $\deg(X) \leqslant 9$, $\deg(Z) \leqslant 9$ or $\deg(V) \leqslant 9$ and $\deg(Z) \leqslant 9$, respectively). We add the edges VX and XY (XY and YZ or VX, XY and YZ, respectively) to H'. The result is a 2-connected spanning subgraph H of G having the required properties.

Subcase 2.2: *Neither X nor Y is of degree* 3 *in G.* In this case H' is a required spanning subgraph of G. ∎

4. POLYHEDRAL MAPS ON COMPACT 2-DIMENSIONAL MANIFOLDS

In this section all manifolds are compact 2-dimensional manifolds. If a graph G is embedded into a manifold $\mathbb{M}$ then the closures of the connected components of $\mathbb{M} - G$ are called the *faces* of G. If each face is a closed 2-cell and each vertex has degree at least three then G is called a *map* in $\mathbb{M}$. If, in addition, no two faces have a multiply connected union then G is called *a polyhedral map* in $\mathbb{M}$. This condition on the union of two faces is equivalent to saying that any two faces that meet, meet on a single vertex or a single edge.

Let in the sequel $\mathbb{S}_g$ ($\mathbb{N}_q$) be an orientable (a non-orientable) compact 2-manifold (also called a surface) of genus g (q, respectively). We recall that the relationship between the Euler characteristic of a surface and its genus is $\chi(\mathbb{S}_g)=2-2g$ and $\chi(\mathbb{N}_q)=2-q$. We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The representativity, rep$(G, \mathbb{M})$, see, e.g., Robertson and Vitray [RoVi] or Mohar [Moh], of an embedded graph G into a compact 2-dimensional manifold $\mathbb{M}$ not homeomorphic to the 2-sphere is the minimum number of intersections of the graph G with a closed (topological) curve $\mathscr{C}$ drawn in $\mathbb{M}$ where $\mathscr{C}$ is taken over all not null-homotopic closed curves. It is routine to show that this minimum can be taken just over all simple closed curves which intersect G in vertices only and which moreover traverse each face of the embedding at most once. Thus the representativity of any polyhedral map on $\mathbb{M}$ is at least three. On the other hand one can easily see (cf. [Moh]) that each 3-connected graph G embedded into a closed 2-manifold $\mathbb{M}$ with rep$(G, \mathbb{M}) \geqslant 3$ forms a polyhedral map on $\mathbb{M}$.

A problem analogous to that one considered in Sections 1 and 2 can also be formulated for polyhedral maps on closed 2-manifolds as follows.

Problem 1. *For a given connected graph H let $\mathscr{G}(\delta, \rho; H, \mathbb{M})$ be a family of all polyhedral maps of minimum vertex degree $\geqslant \delta$ and minimum face size $\geqslant \rho$ on a closed 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic with H. What is the minimum integer $\varphi(\delta, \rho; H, \mathbb{M})$ such that every polyhedral map $G \in \mathscr{G}(\delta, \rho; H, \mathbb{M})$ contains a subgraph K isomorphic to H for which*

$$deg_G(A) \leqslant \varphi(\delta, \rho; H, \mathbb{M}) \qquad \textit{for every vertex } A \in V(K)?$$

(*If such a minimum does not exist we write $\varphi(\delta, \rho; H, \mathbb{M}) = +\infty$.*)

In [JeVo] there is proved

Theorem 5. *Let k be an integer, $k \geqslant 1$, and let $\mathbb{M}$ be a closed 2-manifold with Euler characteristic $\chi(\mathbb{M}) \neq 2$ (this means $\mathbb{M}$ is different from the sphere). Then*

(i) $$2\left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor \leqslant \varphi(3, 3; P_k, \mathbb{M}) \leqslant k \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor,$$

(ii) $$\varphi(3, 3; H, \mathbb{M}) = +\infty \qquad \textit{for any} \quad H \neq P_k.$$

If we follow the ideas of the proof of Theorem 3 and the proof of Theorem 3 in [JeVo] we can prove the following

THEOREM 6. *Let k be an integer, $k \geqslant 1$, and $\mathbb{M}$ be a closed 2-manifold with Euler characteristic $\chi(\mathbb{M}) \neq 2$. Then*

$$\text{(i)}\quad 2\left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor \leqslant \varphi(3,4;P_k,\mathbb{M}) \leqslant k\left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor,$$

$$\text{(ii)}\quad \varphi(3,4,H,\mathbb{M}) = +\infty \qquad \textit{for any} \quad H \ncong P_k.$$

Sketch of the Proof of Theorem 6.

1. The proof of the upper bound differs from that one of Lemma 1 only in using the well known fact that each graph embedded into a closed 2-manifold $\mathbb{M}$ with the Euler characteristic $\chi(\mathbb{M})$ contains a vertex of degree at most $\lfloor (5+\sqrt{49-24\chi(\mathbb{M})})/2 \rfloor$; see, e.g., [Sach]. Because each counterexample G to our upper bound is polyhedral we can use the same tricks as in the proof of Lemma 1.

2. The proof of the lower bounds follows the ideas of the proof of the lower bounds in Lemma 2. Instead of the plane triangulation with minimum degree 5 we use in the orientable case an embedding $R(n)$ of a complete graph K_n with $n = \lfloor (5+\sqrt{49g+1})/2 \rfloor + 1$ into the orientable surface $\mathbb{S}_g$ of genus g, $R(n)$ having representativity $\operatorname{rep}(R_{(n)}, \mathbb{S}_g) \geqslant 2$. In the non-orientable case we use an embedding $Q(n)$ of the complete graph K_n with $n = \lfloor (5+\sqrt{27g+1})/2 \rfloor + 1$, $n \neq 7$, into the non-orientable surface $\mathbb{N}_q$, $Q(n)$ having representativity $\operatorname{rep}(Q(n), \mathbb{N}_q) \geqslant 2$. For $n=7$ we use a 6-regular triangulation of the Klein bottle $\mathbb{N}_2$ described by Thomassen in [Thom]. The existence of the above mentioned embeddings $R(n)$ and $Q(n)$ having the required properties is described in [JeVo].

The construction continues by the decomposition of each triangular face of $R(n)$ and $Q(n)$ in the same way as described in the proof of Lemma 3. One can easily generalize this decomposition also for any r-gonal face, $r \geqslant 4$. Then this decomposition is applied to the faces of $R(n)$ and $Q(n)$ other than triangles. From the resulting map then we delete all original edges of $R(n)$ and $Q(n)$. Denote the obtained map by G. Let $H \in \{R(n), Q(n)\}$. If $\operatorname{rep}(H, \mathbb{M}) \geqslant 3$ then clearly $\operatorname{rep}(G, \mathbb{M}) \geqslant 3$. If $\operatorname{rep}(H, \mathbb{M}) = 2$ then any topological cycle which passes through exactly two vertices A and B of H must cross the graph G at least 3 times. The reason is that if not, then in H, there is an r-gonal face α, $r \geqslant 4$, incident with A and B on which these vertices are not adjacent. But the above described decomposition of α separates the vertices A and B in α. The deletion of the original edges of

H does not have influence on the representativity. So we have $\operatorname{rep}(G, \mathbb{M}) \geqslant 3$. This, because G is 3-connected, means that G is polyhedral on $\mathbb{M}$. One can easily check that each k-path P_k in G contains a vertex of degree at least $\lfloor k/2 \rfloor \lfloor (5+\sqrt{49-24\chi(\mathbb{M})})/2 \rfloor$.

3. To prove that $\varphi(3, 4; H, \mathbb{M}) = +\infty$ for $H \not\cong P_k$, $k \geqslant 1$, we proceed as in Lemma 3. Let H and an integer $m \geqslant 3$ be given. First the graph H is packed into a triangulation T of $\mathbb{M}$. Then into each triangular face of T we insert the same structure as described in Lemma 3. ∎

5. REMARKS

1. In Theorems 1, 2 and 3 all light graphs in the families $\mathscr{P}(3, 3)$, $\mathscr{P}(4, 3)$ and $\mathscr{P}(3, 4)$, respectively, are characterized. The situation changes radically in the families $\mathscr{P}(5, 3)$ and $\mathscr{P}(3, 5)$. From Theorem 1 it follows that each path P_k, $k \geqslant 1$, is also light in both families, $\mathscr{P}(5, 3)$ and $\mathscr{P}(3, 5)$. But from the classical result of Lebesgue [Lebe] it follows that a 3-cycle C_3 is light in $\mathscr{P}(5, 3)$ and a 5-cycle C_5 is light in $\mathscr{P}(3, 5)$. Recently, Jendrol' and Madaras [JeMa] proved that the star $K_{1,r}$, $r \geqslant 3$, is light in $\mathscr{P}(5, 3)$ if and only if $r \in \{3, 4\}$. In [JMST] Jendrol' *et al.* proved that no graph H with $\Delta(H) \geqslant 6$ or with a block on at least 11 vertices is light in $\mathscr{P}(5, 3)$. Moreover they characterized, in [JMST], all light cycles in the family $\mathscr{T}(5)$ of all plane triangulations of minimum degree 5. However, the problem of characterization of all light graphs in $\mathscr{P}(5, 3)$ and in $\mathscr{P}(3, 5)$ remains still open.

2. For the graph H and the family $\mathscr{H}$ defined in the introduction one can also introduce the value $w(H, \mathscr{H})$, see [JHST], the smallest integer with the property that every graph $G \in \mathscr{H}$, which has a subgraph isomorphic to H, contains also a subgraph K, $K \cong H$, such that

$$w_G(K) = \sum_{A \in V(K)} \deg_G(A) \leqslant w(H, \mathscr{H}).$$

The sum $w_G(K)$ is called the *weight* of K in the graph G; see, e.g., [Grü].

So one can define $w(K, \mathscr{H})$ to be the *weight* of H in the family $\mathscr{H}$ and can say that the graph H *is light* in $\mathscr{H}$ if $w(H, \mathscr{H}) < +\infty$. As $w(H, \mathscr{H}) < \infty$ if and only if $\varphi(H, \mathscr{H}) < +\infty$, both above definitions of a light graph are equivalent. We prefer to use the first one because it simplifies things.

3. The precise value of $w(\delta, \rho, H)$ is known only for a few small light graphs. For a recent survey see [JMST]. This leads to the following

Problem 2. *Determine the value* $w(\delta, \rho; P_k)$ *for each* $k \geqslant 3$ *and each pair* $(\delta, \rho) \in \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}$.

The known results concerning this problem are the following:

Kotzig [Kot1]	$w(3, 3; P_2) = 13$ and $w(4, 3; P_2) = 11$.
Wernicke [Wern]	$w(3, 5; P_2) = 11$.
Lebesgue [Lebe]	$w(3, 4; P_2) = 8$, $w(3, 5; P_2) = 6$, $w(3, 5; P_3) = 9$, $w(3, 5; P_4) = 12$, $w(3, 5; P_5) = 17$ and $w(3, 4; P_3) \leqslant 12$.
Ando *et al.* [AIK]	$w(3, 3; P_3) = 21$.
Borodin [Boro]	$w(4, 3; P_3) = 17$.
Franklin [Fran]	$w(5, 3; P_3) = 17$.
Fabrici and Jendrol' [FaJe1, FaJe2]	$k \log_2 k \leqslant w(3, 3; P_k) \leqslant 5k^2$, $k \geqslant 4$.
Fabrici *et al.* [FHJW]	$w(5, 3; P_k) \leqslant w(4, 3; P_k) \leqslant 5k^2 - 7k$, $k \geqslant 8$.
Jendrol' and Owens [JeOw]	$w(3, 5; P_k) \leqslant \frac{5}{3}k^2$.

From Theorem 3 we have $w(3, 4; P_k) \leqslant \frac{5}{2}k^2$. For recent progress concerning 3-paths in $\mathscr{P}(3, 3)$; see [Jen] and [Boro].

ACKNOWLEDGMENTS

Support by Slovak VEGA Grant 1/4377/97 is acknowledged. This work was supported by the Deutscher Akademischer Austauschdienst (DAAD). The authors are grateful to the referee for many helpful suggestions.

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